Bi-invariant metrics on groups

Jarek Kędra

University of Aberdeen

Selected Topics in Mathematics - 4 OCTOBER 2024

In collaboration with: Brandenbursky, Gal, Jaspars, Karlhofer, Libman, Marcinkowski, Martin, Shelukhin, Trost.

Overview

- Definition.
- Riemannian Geometry.
- Hamiltonian dynamics.
- Group Theory.
- Biology.
- General outlook.
- Free groups.

“It is impossible to understand an unmotivated definition. . . “

Definition

Let G be a group. A metric d on G is called ***bi-invariant*** if both the multiplication from the right and from the left are isometries:

$$d(xg, yg) = d(x, y) = d(gx, gy)$$

for all $x, y, g \in G$.

“It is impossible to understand an unmotivated definition. . . “

Definition

Let G be a group. A metric d on G is called ***bi-invariant*** if both the multiplication from the right and from the left are isometries:

$$d(xg, yg) = d(x, y) = d(gx, gy)$$

for all $x, y, g \in G$.

$$\implies d\left(g^{-1}xg, 1\right) = d(xg, g) = d(x, 1)$$

"It is impossible to understand an unmotivated definition. . . "

Definition

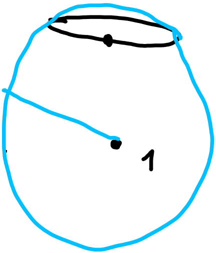
Let G be a group. A metric d on G is called ***bi-invariant*** if both the multiplication from the right and from the left are isometries:

$$d(xg, yg) = d(x, y) = d(gx, gy)$$

for all $x, y, g \in G$.

$\implies d\left(g^{-1}xg, 1\right) = d(xg, g) = d(x, 1)$

$\implies$ Conjugacy classes live on spheres centered at 1.



“It is impossible to understand an unmotivated definition. . . “

Definition

Let G be a group. A metric d on G is called ***bi-invariant*** if both the multiplication from the right and from the left are isometries:

$$d(xg, yg) = d(x, y) = d(gx, gy)$$

for all $x, y, g \in G$.

$\Longrightarrow$ $d\left(g^{-1}xg, 1\right) = d(xg, g) = d(x, 1)$

$\Longrightarrow$ Conjugacy classes live on spheres centered at 1.

- $\|g\| = d(g, 1)$ – a conjugation-invariant norm.
- $d(g, h) = \|gh^{-1}\| = \|g^{-1}h\|$.

$d \longleftrightarrow \|\ \|$

Riemannian geometry

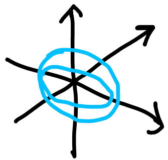
- G a Lie group with Lie algebra $\mathfrak{g}$.
- Choose an inner product on $\boxed{\mathfrak{g} = T_1 G.}$
- Propagate over TG with left multiplication:

$$\langle X, Y \rangle_g = \langle dL_{g^{-1}} X, dL_{g^{-1}} X \rangle_1.$$

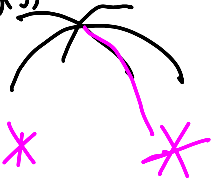
$\Longrightarrow$ $\boxed{\text{Left-invariant}}$ metric on G.

Riemannian geometry

$$G = \mathrm{SO}(3)$$
$$\mathrm{Ad}\colon\ \mathrm{SO}(3) \to \mathrm{Aut}(\mathrm{so}(3))$$

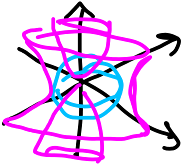


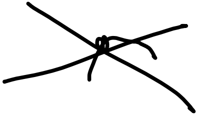
SO(3)



COMPACT

$$G = \mathrm{SL}(2, \mathbf{R})$$
$$\mathrm{Ad}\colon\ \mathrm{SL}(2, \mathbf{R}) \to \mathrm{Aut}(\mathrm{sl}(2, \mathbf{R}))$$





SIMPLE
NON-COMPACT

Hamiltonian dynamics

- (M, ω) - a symplectic manifold.

$\iff$ ω - closed $\boxed{\text{non-degenerated two-form}}$

$\implies$ Isomorphism between vector fields and one-forms:

1-FORM

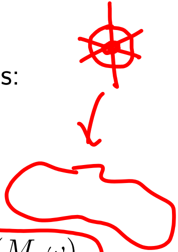
$$X \overset{\cong}{\longleftrightarrow} \omega(X, -) = \iota_X \omega$$

Hamiltonian dynamics

- (M, ω) - a symplectic manifold.

$\Longleftrightarrow$ ω - closed non-degenerated two-form

$\Longrightarrow$ Isomorphism between vector fields and one-forms:



$$X \longleftrightarrow \omega(X, -) = \iota_X \omega$$

$\varphi^* \omega = \omega$

• Group	$\mathrm{Diff}(M)$	$\supseteq$	$\mathrm{Symp}(M, \omega)$	$\supseteq$	$\boxed{\mathrm{Ham}(M, \omega)}$
• Algebra	X		$\boxed{L_X \omega = 0}$		$\iota_X \omega = dH$

$\boxed{\text{VECT F.}}$

$$\parallel$$

$$\boxed{d \boxed{\iota_X \omega}} + \cancel{\iota_X d\omega}$$

CLOSED 1 FORM

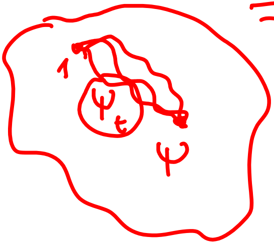
$C^\infty(M)/\mathbb{R}$

$\mathrm{osc}\, H = \max H - \min H$

Hofer's metric

HOFER'S NORM

$$\|\psi\| = \inf_H \int_0^1 \operatorname{osc} H(t,-)\; dt$$



$$d_H(\psi,\varphi) = \|\psi \cdot \varphi^{-1}\|$$

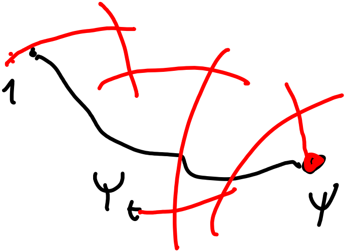
$$\psi_t \sim X_t \sim i_{X_t}\omega = dH_t$$

All known examples have infinite Hofer diameter.

Autonomous metric

- $\psi \in \mathrm{Ham}(M,\omega)$.
- $\psi=\psi_1$, where $\{\psi_t\}\in \mathrm{Ham}(M,\omega)$.
- $\psi_t \quad \longleftrightarrow \quad X_t \quad \longleftrightarrow \quad H_t$.
- ψ is **autonomous** if $H_t = H$ is time independent.

$$\|\psi\|_{Aut} = \min\{n\in\mathbf{N} \mid \psi=\alpha_1\cdots\alpha_n,\ \alpha_i \text{ is autonomous}\}$$



$= 7$

THM (BRANDENBURSKY-K)
$\mathrm{Ham}^c(\mathbb{R}^{2n},\omega_0)$
AUTONOMOUS DIAMETER IS EITHER 2 OR $\boxed{3}$.

OPEN: WHICH ONE?

Group theory

G a group generated by $S \subseteq G$; $S = S^{-1}$. Word norm and metric:

- $\|g\|_S = \min\{n \in \mathbf{N} \mid g = s_1 \cdots s_n,\ s_i \in S\}$ ← WORD NORM
- $d_S(g,h) = \|gh^{-1}\|_S$ ← `right-invariant`

If $\boxed{g^{-1}Sg = S}$ for every $g \in G$ then the norm is conjugation-invariant and the metric bi-invariant.

Examples

- Coxeter groups.
  - S – the set of all reflections.
  - d_S – the reflection metric is **bi-invariant**.

Examples

- Coxeter groups.
  - S – the set of all reflections.
  - d_S – the reflection metric is **bi-invariant**.
- Finitely generated groups.

GGT

  - S - finite generating set.
  - d_S - **right-invariant** unless G is (virtually) abelian.
  - Up to Lipschitz equivalence, d_S does not depend on S.

Examples

- Coxeter groups.
  - S – the set of all reflections.
  - d_S – the reflection metric is **bi-invariant**.
- Finitely generated groups.
  - S - finite generating set.
  - d_S - **right-invariant** unless G is (virtually) abelian.
  - Up to Lipschitz equivalence, d_S does not depend on S.
- Commutator subgroup $[G, G] \subseteq G$.
  - S - the set of all commutators $[g, h] \in [G, G]$, $g, h \in G$.
  - d_S – commutator metric is **bi-invariant**.
  - Good topological interpretation.

scl
CALEGARI

Examples

- Coxeter groups.
  - S – the set of all reflections.
  - d_S – the reflection metric is **bi-invariant**.
- Finitely generated groups.
  - S - finite generating set.
  - d_S - **right-invariant** unless G is (virtually) abelian.
  - Up to Lipschitz equivalence, d_S does not depend on S.
- Commutator subgroup $[G, G] \subseteq G$.
  - S - the set of all commutators $[g, h] \in [G, G]$, $g, h \in G$.
  - d_S – commutator metric is **bi-invariant**.
  - Good topological interpretation.
- $\mathbf{F}_2 = \langle a, b \rangle$ - free group on two generators.
  - S – all conjugates of a, b and their inverses.
  - d_S – ***bi-invariant***.
  - Good algorithms for computations (we have a software [2013]).

$\|g\|_S$

Examples

$$g^{-1} p g = g^{-1} \cdot \boxed{r(g^{-1}) \cdot r(g)} \cdot p g$$

- $\mathbf{F}_n$.
  - $\boxed{S} \subseteq \mathbf{F}_n$ – all palindromes.
  - Palindromic metric.
  - Infinite diameter (at the end on request).

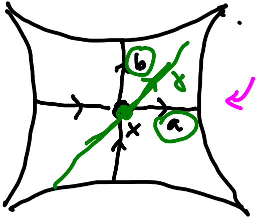
Examples

- $\mathbf{F}_n$.
  - $S \subseteq \mathbf{F}_n$ – all palindromes.
  - Palindromic metric.
  - Infinite diameter (at the end on request).
- (M, g) - Riemannian manifold.
  - $S \subseteq \pi_1(M, x)$ – the set of all closed geodesics. THROUGH x
  - Assume S generates $\pi_1(M, x)$.

$\Longrightarrow$ the closed geodesic metric on $\pi_1(M, x)$.

$\pi_1 = F_2$

EX $T^2 \setminus \{p\}$



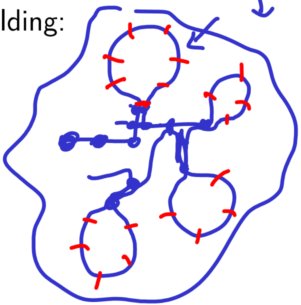
$I(\gamma) = \gamma^{-1}$

$I(w(a,b)) = \underbrace{w(a^{-1}, b^{-1})} = w(a,b)^{-1}$

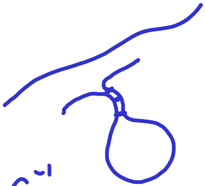
IFF $w(a,b)$ is A PALINDROME

RNA folding

- RNA: sequence of letters A,C,G,U.
- Pairings: A-U, C-G.
- Folding:



$A - A^{-1}$ $\quad$ $C - C^{-1}$



RNA folding

- RNA: sequence of letters A,C,G,U.
- Pairings: A-U, C-G.
- Folding:
- It is the conjugation-invariant word norm on $\mathbf{F}_2$!

RNA folding

- RNA: sequence of letters A,C,G,U.
- Pairings: A-U, C-G.
- Folding:
- It is the conjugation-invariant word norm on $\mathbf{F}_2$!
- Biologists found our algorithm in 1980!

Bi-invariant word metrics

$S \subseteq G \quad \langle \bigcup_{g \in G} g^{-1} S g \rangle = G$

- $\boxed{G}$ – normally finitely generated group.
- S – union of finitely many conjugacy classes (and inverses).
- d_S – **the** bi-invariant word metric.

$(G, d_S) \xrightarrow{L} (G, d)$

EX $\quad G$ - simple

$S = \mathrm{Conj}(g^{\pm 1})$

Bi-invariant word metrics

- G – normally finitely generated group.
- S – union of finitely many conjugacy classes (and inverses).
- d_S – **the** bi-invariant word metric.
- Q: Is the diameter of d_S finite or infinite?
  - $\mathbf{F}_n$ – infinite.
  - $\mathrm{SL}(n, \mathbf{Z})$ – finite if $n \geq 3$.
  - $\mathrm{SO}(5, \mathbf{Z}[1/5])$ – ?
  - $\mathrm{Diff}_0(S^n)$ – finite.
  - $\mathrm{Diff}_0(\Sigma_g)$ – infinite if $g \geq 2$.
  - $\mathrm{Diff}_0(S^n, \mathrm{vol})$ - ?, $n \geq 3$.
  - Etc. Ask questions at the end.

`Non-el. hyperbolic`

`Chevalley`

→ `Cocompact lattice`

S^n M dim $\neq 2, 4$

?

The free group

- $G = \mathbf{F}_2 = \langle a, b \rangle$
- $g = b^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}a^{-1}baba$

The free group

? $\|g\| > \|g^k\|$?

- $G = \mathbf{F}_2 = \langle a, b \rangle$
- $g = b^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}a^{-1}baba$
- $\|g^2\| = \|g\| = 4$

Biological meaning?

The free group

- $G = \mathbf{F}_2 = \langle a, b \rangle$
- $g = b^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}a^{-1}baba$
- $\|g^2\| = \|g\| = 4$ `Biological meaning?`
- $\|g^n\| = 4, 4, 6, 8, 10, 10, 12, 14, 16, 16, 18, 20, 22, 22, 24, 26, \cdots$
- Hmmm...this is quite regular...

The free group

- $G = \mathbf{F}_2 = \langle a, b \rangle$
- $g = b^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}a^{-1}baba$
- $\|g^2\| = \|g\| = 4$ `Biological meaning?`
- $\|g^n\| = 4, 4, 6, 8, 10, 10, 12, 14, 16, 16, 18, 20, 22, 22, 24, 26, \cdots$
- Hmmm... this is quite regular...
- $\|[a, b]^n\| = 2, 4, 4, 6, 6, 8, 8, \cdots = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n+2 & \text{if } n \text{ is even} \end{cases}$
- In fact...

Jaspars's Theorem

Let $g \in \mathbf{F}_n$ be any element in a free group. Then the sequence $\|g^n\|$ is *uniformly semi-arithmetic*. In particular, the limit

$$\lim_{n\to\infty} \frac{\|g^n\|}{n}$$

WHICH EXISTS BY FEKETE

is rational.

CALEGARI

scl IS RAT ON FREE GROUPS

THANK YOU! Any questions?

$SL(3,\mathbb{Z})$ – BOUNDED GENERATION

$\|SL(3;\mathbb{Z})\| \leq 72$

$\boxed{E_1 \cdot \ldots \cdot E_{36}}$

WRT S.

$\|E_i\| \leq 2$

DIAM IS FINITE ~~WRT~~ EVER FINITE NORM. GEN. SET.

? HOW DOES IT DEP. ON THE CHOICE?